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POST-BUCKLING BEHAVIOR OF A CLOSED SPHERICAL SHELL

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M. Iu. ZHUKOV and L.S. SRUBSHCHIK
(Rostov-on-Don)
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The question of new equilibrium modes of a uniformly compresses closed elastic spherical shell for loading values close to the critical one is considered for which the membrane state of stress loses stability. The problem [1] reduces to constructing solutions branching off from the trivial solution in the neighborhood of the bifurcation point, for the equations in [2]. The investigation is carried out by the Liapunov-Schmidt method for a broad class of operator equations in Banach space [3].

The author of [4, 5] used the analytical Liapunov-Schmidt method earlier to construct new equilibrium modes in the case of plates and shallow shells. The problem of the bifurcation of the trivial solution of a shallow spherical segment by the Poincaré method was investigated in [6], where meridian stress resultants in equilibrium with the uniformly distributed surface pressure are given on the edge, whereupon a membrane equilibrium mode always exists. For the problem of an uniformly compressed closed sphere when the spectrum is simple, the behavior of the solutions in the neighborhood of the bifurcation point has been studied in [7] numerically on a computer by using the method of "adjustment". The survey [8] is devoted to this same problem.

1. Formulation of the problem. The Reissner equations for axisymmetric elastic deformation of a closed spherical shell subjected to uniformly distributed pressure [2] are considered in dimensionless form

$$\varepsilon^2 \Big\{(\Phi - \Phi_0)'' + \operatorname{ctg} \xi\, (\Phi - \Phi_0)' - \frac{\cos\Phi}{\sin^2\xi}(\sin\Phi - \sin\Phi_0) + \tag{1.1}$$
$$+ \frac{\nu\Phi_0'}{\sin\xi}(\cos\Phi - \cos\Phi_0)\Big\} = \frac{1}{\sin\xi}\,(N\sin\Phi - T\cos\Phi)$$
$$\Big\{N'' + \operatorname{ctg}\xi\, N' - \Big(\frac{\cos^2\Phi_0}{\sin^2\xi} - \nu\Phi_0'\,\frac{\sin\Phi_0}{\sin\xi}\Big) N\Big\} = \frac{1}{\sin\xi}\,\{\cos\Phi - \cos\Phi_0 +$$

$$+\nu \sin \Phi_0 T' - (\sin^2 \xi p)'\} + \left[\frac{\cos \Phi_0 \sin \Phi_0}{\sin^2 \xi} + \nu \Phi_0' \frac{\cos \Phi_0}{\sin \xi}\right] T - \nu p \cos \Phi_0$$

with the boundary conditions

$$N(0) = N(\pi) = 0, \qquad \Phi(0) = 0, \qquad \Phi(\pi) = \pi \tag{1.2}$$

Equations (1.1), (1.2) are obtained from (28), (29) in [2] if the influence of the transverse shear is neglected and the following relationships are used

$$\Psi_H = NR^2\gamma E\varepsilon, \quad T = \frac{\Psi_V}{R^2\gamma E\varepsilon} = -\frac{1}{R^2}\int_0^\xi q \sin \xi d\xi \quad \Phi_0 = \xi, \quad \alpha = R$$

$$p_H = pE\gamma\varepsilon, \quad p_V = qE\gamma\varepsilon, \quad r = R\sin \xi, \quad z = -R\cos \xi$$

$$\varepsilon = h/R\gamma, \quad \gamma^2 = 12(1-\nu^2), \quad p = -\rho \sin \xi, \quad q = \rho \cos \xi \tag{1.3}$$

Here $\Phi(\xi)$ is the angle which a shell element makes with the horizontal axis after deformation at the point corresponding to ξ; Ψ_H, Ψ_V and p_H, p_V are, respectively, the horizontal and vertical stress and loading components, E is the Young modulus, ν the Poisson ratio, $h = \text{const}$ is the shell thickness, and R the radius of the sphere.

The problem (1.1), (1.2) has a trivial solution corresponding to the membrane state of stress for any values of the loading ρ

$$\Phi = \xi, \qquad N = -{}^1/_4\, \rho \sin 2\xi \qquad (0 \leqslant \xi \leqslant \pi)$$

Hence, it is convenient to make the substitution

$$\Phi = \beta + \xi, \qquad N = \psi - {}^1/_4\, \rho \sin 2\xi$$

Since the new equilibrium modes are sought close to the membrane mode, it is then natural to assume the quantity β small and to neglect terms above the second order in β in (1.1), (1.2). We hence obtain the equations

$$\varepsilon^2\{A\beta + (1-\nu)\beta\} + {}^1/_2\, \rho\beta - \psi = {}^1/_2\, \varepsilon^2(\nu - 3)\beta^2 \operatorname{ctg}\xi + \psi\beta \operatorname{ctg}\xi -$$

$$- \{A\psi + (1+\nu)\psi\} - \beta = {}^1/_2\, \beta^2 \operatorname{ctg}\xi$$

$$Aq \equiv q'' + \operatorname{ctg}\xi\, q' - \sin^{-2}\xi\, q \qquad (q = \beta, \psi) \tag{1.4}$$

with the boundary conditions

$$\beta(0) = \beta(\pi) = \psi(0) = \psi(\pi) = 0 \tag{1.5}$$

A number of the solutions branching off in the neighborhood of the bifurcation points of the problem (1.4), (1.5) is determined below, and asymptotic expansions are constructed for each of them (Theorems 1 - 3). This number will fluctuate between zero and three as a function of the multiplicity of the spectrum of the linearized problem.

Let us introduce the Hilbert spaces:

1) The space E_1 consisting of the closure of the set of smooth vector functions $x \equiv (\beta, \psi)$ satisfying the boundary conditions (1.5) with the finite norm

$$\|x\|_{E_1}^2 = \int_0^\pi \sin \xi\, [(A\beta)^2 + (A\psi)^2]\, d\xi$$

2) The space E_2 of vector functions $u \equiv (u_1, u_2)$ with the finite norm

$$\| u \|_{E_2}^2 = \int_0^\pi \sin\xi\, [u_1^2 + u_2^2]\, d\xi$$

Then the problem (1.4), (1.5) can be written in the form of the functional equation

$$Bx = D(x) \tag{1.6}$$

Here B is a linear, and $D(x)$ quadratic operator from E_1 and E_2

$$B = \left\| \begin{matrix} \varepsilon^2 \{A + 1 - \nu\} + {}^1/_2 \rho, & -1 \\ -1 & , \quad -\{A + 1 + \nu\} \end{matrix} \right\|$$

$$D(x) = \left\| \begin{matrix} {}^1/_2 \varepsilon^2 (\nu - 3) \beta^2 \operatorname{ctg}\xi + \psi\beta \operatorname{ctg}\xi \\ {}^1/_2 \beta^2 \operatorname{ctg}\xi \end{matrix} \right\|$$

The possibility of writing the problem (1.4), (1.5) in the form (1.6) follows from the estimates

$$\max_{0 \leqslant \xi \leqslant \pi} |\beta| \leqslant M \| A\beta \|_{E_2}, \qquad \int_0^\pi \frac{\cos^2\xi}{\sin\xi} \beta^4 d\xi \leqslant M \| A\beta \|_{E_2}^2$$

These estimates follow from the analysis of the problem

$$A\beta = f, \qquad \beta(0) = \beta(\pi) = 0$$

written in entegral form, and the application of the Cauchy-Buniakowski inequality.

2. Investigation of the linearized problem. Let us consider the linearized problem

$$Bx = 0 \tag{2.1}$$

in the neighborhood of the trivial solution $x = (0, 0)$ of (1.6). The eigenfunctions of the problem (2.1) are sought in the form

$$\beta = \sum_{k=1}^{\infty} a_k P_k^1 (\cos\xi), \qquad \psi = \sum_{k=1}^{\infty} b_k P_k^1 (\cos\xi) \tag{2.2}$$

Here $P_k^1(\cos\xi)$ are the associated Legendre polynomials forming a complete orthonormal system [9]. To determine a_m, b_m we substitute (2.2) into (2.1), multiply by $P_m^1(\cos\xi)$, integrate with weight $\sin\xi$ between 0 and π taking account of the relationship

$$AP_k^1(\cos\xi) = -k(k+1) P_k^1(\cos\xi) \tag{2.3}$$

and obtain the system

$$\begin{aligned} &\{\varepsilon^2 [-m(m+1) + 1 - \nu] + {}^1/_2 \rho\} a_m - b_m = 0 \\ &-a_m - [-m(m+1) + 1 + \nu] b_m = 0 \quad (m = 1, 2, \ldots) \end{aligned} \tag{2.4}$$

Equating the determinant of the system (2.4) to zero for each m, we obtain the eigenvalues of the problem (2.1)

$$\rho = 2\varepsilon^2 (m^2 + m - 1 + \nu) + 2/(m^2 + m - 1 - \nu) \qquad (m = 1,2,\ldots) \qquad (2.5)$$

We find the least eigenvalue

$$\rho_* = \min(\rho_{[l]}, \rho_{[l]+1}) = 4\varepsilon + 4\nu\varepsilon^2 \qquad (2.6)$$

for fixed ε and ν . Here $[l]$ denotes the integral part of the number l satisfying the relationship

$$l^2 + l = \varepsilon^{-1} + 1 + \nu, \qquad l > 0$$

It is seen that the eigennumbers ρ_* can be simple and double-valued; this latter holds if $\rho_{[l]} = \rho_{[l]+1}$. If the eigenvalue ρ_* is simple and $\rho_{[l]} < \rho_{[l]+1}$, then its corresponding eigenfunction of the operator B is

$$\varphi_* = \varphi_k = \begin{Vmatrix} a_k P_k^1(\cos\xi) \\ b_k P_k^1(\cos\xi) \end{Vmatrix} \qquad \begin{matrix} k = [l] \\ a_k = b_k(k^2 + k - 1 - \nu) \end{matrix} \qquad (2.7)$$

If ρ_* is double-valued, then its two corresponding eigenfunctions are φ_k and φ_{k+1}.

3. Construction of new equilibrium modes. Let us utilize the results of [3] to construct new equilibrium modes.

Theorem 1. Let ρ_k be a simple eigenvalue of the problem (2.1), then:

(a) If k is even, then in each of the half-neighborhoods $(\rho_k - \delta, \rho_k)$ and $(\rho_k, \rho_k + \delta)$ one new solution branches off;

(b) If k is odd, then no new solutions originate in one of the half-neighborhoods and two solutions appear in the other.

Proof. Let ρ_0 denote the eigenvalue of the problem (2.1) and λ a small parameter. Then assuming

$$\rho = \rho_0 + \lambda, \qquad |\lambda| < \delta$$

(1.6) can be written as

$$B_0 x = \lambda C x + D(x), \qquad C = \begin{Vmatrix} -{}^1/_2, & 0 \\ 0, & 0 \end{Vmatrix} \qquad (3.1)$$

Here B_0 is the operator B in which ρ is replaced by the eigenvalue ρ_0. Let us seek small solutions $x(\lambda)$ of (3.1). The operator B_0 is found in the spectrum, and hence has no inverse. Let us construct the operator B_1

$$B_1 x \equiv B_0 x + \sum_{i=1}^{n} (x, \gamma_i)_{E_1} z_i$$

Here γ_i, z_i have been determined from [3]. According to the generalized Schmidt lemma, there exists an inverse linear bounded operator $\Gamma = B_1^{-1}$. It is easy to see that B_0 is a self-conjugate operator, and we can hence put $\psi_i = \varphi_i$, where ψ_i is the eigenvector of the conjugate operator. Let us take φ_i as γ_i then to determine a_i, b_i the equation

$$(a_i^2 + b_i^2)\, i^2 (i + 1)^2 = 1 \qquad (3.2)$$

is added to the second equation (2.4). Let us take the vector z_i in the form $z_i = i^2 (i + 1)^2 \varphi_i$. When ρ_k is a simple eigenvalue, we replace (3.1) by the equivalent system

$$B_1x = \lambda Cx + D(x) + \mu z_k, \qquad \mu = (x, \gamma_k)_{E_1} \tag{3.3}$$

Let us seek the small solution $x = x(\lambda, \mu)$ of this system in the form of the series

$$x = \sum_{i=1}^{\infty} x_{i0}\mu^i + \sum_{i=0}^{\infty}\mu^i \sum_{j=1}^{\infty} x_{ij}\lambda^j, \quad x_{ij} \equiv (\beta_{ij}(\xi), \psi_{ij}(\xi)) \tag{3.4}$$

Substituting (3.4) into the operator $D(x)$, we obtain

$$D(x) = \sum_{i+j\geqslant 2} D_{ij}\mu^i\lambda^j \tag{3.5}$$

$$D_{ij} = \sum_{\substack{m+p=i\\ t+k=j}} \alpha \left\| \begin{matrix} \{\varepsilon^2(\nu-3)\beta_{mt}\beta_{pk} + \beta_{mt}\psi_{pk} + \beta_{pk}\psi_{mt}\} \operatorname{ctg}\xi \\ \beta_{mt}\beta_{pk}\operatorname{ctg}\xi \end{matrix} \right\|$$

Here $\alpha = 1/2$ if $m = p$, $t = k$, and $\alpha = 1$ in the remaining cases. Substituting (3.4) in the first equation of (3.3), taking into account (3.5) and equating terms in powers of $\mu^i\lambda^j$,, we obtain the recurrent system to find x_{ij}

$$B_1x_{ij} = f_{ij},\ f_{01} = 0,\ f_{11} = Cx_{10} + D_{11},\ f_{10} = z_k \tag{3.6}$$
$$f_{20} = D_{20},\ f_{30} = D_{30},\ldots$$

Then substituting (3.4) in the second equation of the system (3.3), we obtain the bifurcation equation

$$\sum_{k=2}^{\infty} L_{k0}\mu^k + \sum_{k=0}^{\infty} \mu^k \sum_{l=1}^{\infty} L_{kl}\lambda^l = 0$$
$$L_{ij} = (x_{ij}, \gamma_k)_{E_1} = (B_1x_{ij}, \psi_k)_{E_2} \tag{3.7}$$

We find all the x_i from (3.6) taking account of (3.5):

$$x_{01} = 0,\ x_{10} = \varphi_k,\ x_{11} = \Gamma f_{11},\ x_{20} = \Gamma f_{20}$$

We now have for the L_{ij} from (3.7)

$$L_{01} = 0,\ L_{11} = (Cx_{10}, \psi_k)_{E_2},\ L_{20} = (f_{20}, \psi_k)_{E_2},\ L_{30} = (f_{30}, \psi_k)_{E_2}$$

Let us write the coefficients L_{ij} by utilizing the form $\varphi_k, \psi_k, \gamma_k, z_k$

$$L_{11} = -\frac{1}{2}a_k^2\int_0^\pi P_k^1(\cos\xi)P_k^1(\cos\xi)\sin\xi\,d\xi = -\frac{1}{2}a_k^2 < 0$$

$$L_{20} = \frac{1}{2}\int_0^\pi \{\varepsilon^2(\nu-3)a_k^3 + 3a_k^2b_k\}[P_k^1(\cos\xi)]^3\cos\xi\,d\xi$$

$$L_{30} = \int_0^\pi \{[\varepsilon^2(\nu-3)\beta_{10}\beta_{20} + \beta_{20}\psi_{10} + \beta_{10}\psi_{20}]\beta_{10} + \beta_{10}\beta_{20}\psi_{10}\}\cos\xi\,d\xi \tag{3.8}$$

Here the a_k, b_k have been determined from (2.4), (3.2). It can be shown that

$$S_{jk} \equiv \int_0^{\pi} [P_k^1(\cos\xi)]^2 P_j^1(\cos\xi)\cos\xi\, d\xi = \begin{cases} 0, & \text{if } j \text{ is odd} \\ 0, & \text{if } j > 2k \\ \neq 0, & \text{if } j \text{ is even and } j \leqslant 2k \end{cases} \qquad (3.9)$$

Now let k be even, then it follows from (3.8), (3.9) that $L_{20} \neq 0$, $L_{11} < 0$, and the bifurcation equation (3.7) has the form

$$L_{20}\mu^2 + L_{11}\lambda + \ldots = 0$$

Hence

$$\mu = -L_{11} / L_{20}^{-1}\lambda + o(\lambda)$$

The asymptotic representation of the solution (3.4) is written as

$$x = -L_{11}L_{20}^{-1}x_{10}\lambda + L_{11}^2L_{20}^{-2}x_{20}\lambda^2 - L_{11}L_{20}^{-1}x_{11}\lambda^2 \qquad (|\lambda| < \delta) \qquad (3.10)$$

Then one small solution originates in the half-neighborhood $(\rho_k - \delta,\ \rho_k)$ and $(\rho_k,\ \rho_k + \delta)$. Let k be odd, then if follows from (3.8), (3.9) that $L_{20} = 0$, $L_{30} \neq 0$, $L_{11} < 0$, and the bifurcation equation (3.8) becomes

$$L_{30}\mu^3 + L_{11}\mu\lambda + (*) = 0$$

Hence

$$\mu = \pm(-L_{11}L_{30}^{-1}\ \lambda)^{1/2} + o\ (\lambda^{1/2})$$

In this case the asymptotic representation of the solution (3.4) is

$$x = \pm(-L_{11}L_{30}^{-1}\lambda)^{1/2}x_{10} - L_{11}L_{30}^{-1}x_{20}\lambda \pm (-L_{11}L_{30}^{-1}\lambda)^{1/2}x_{11}\lambda \quad (|\lambda| < \delta) \qquad (3.11)$$

Depending on the sign of the ratio $-L_{11} / L_{30}$ no new solutions will originate in one of the half-neighborhood $(\rho_k - \delta, \rho_k)$ and $(\rho_k,\ \rho_k + \delta)$, and two new solutions will appear in the other one. In the case of the minimum eigenvalue $\rho_k = \rho_*$, it can be indicated in precisely which half-neighborhood the pair of new solutions originates.

Theorem 2. Let $\rho_k = \rho_*$ and let k be odd, then two new solutions originate in the half-neighborhood ($\rho_k - \delta,\ \rho_k$) .

Proof. We obtain $L_{30} < 0$ from (3.8) and the Courant minimax principle [10] by using (3.5), (3.6). The solution (3.11) is real if $-L_{11}L_{30}^{-1}\ \lambda \geqslant 0$, i.e. $\lambda \leqslant 0$.

Theorem 3. If the minimum eigenvalue ρ_k of the problem (2.1) is double-valued, then three new solutions will originate in each of the half-neighborhoods $(\rho_k - \delta,\ \rho_k)$ and $(\rho_k,\ \rho_k + \delta)$.

Proof. Let $\rho_0 = \rho_k = \rho_j$ be double-valued. This holds when

$$\varepsilon^2 = [(k^2 + k - 1 - \nu)\ (j^2 + j - 1 - \nu)]^{-1}$$

Let us replace (3.1) by the equivalent system

$$B_1x = \lambda Cx + D\ (x) + \mu_1z_j + \mu_2z_k,\ \mu_1 = (x,\ \gamma_j)_{E_1},\ \mu_2 = (x,\ \gamma_k)_{E_1} \qquad (3.12)$$

Let us seek small solutions $x = x\ (\lambda,\ \mu_1,\ \mu_2)$ of this system in the form of the series

$$x = x_{001}\lambda + x_{100}\mu_1 + x_{010}\mu_2 + \sum_{i+n+m\geqslant 2} x_{inm}\mu_1{}^i\mu_2{}^n\lambda^m \tag{3.13}$$

$$x_{inm} \equiv (\beta_{inm}(\xi), \qquad \psi_{inm}(\xi))$$

Let us assume that j and k have different parity (precisely such a case is possible for the least eigenvalue). For definiteness, let j be odd. Analogously to the preceding, we obtain the bifurcation equation

$$\Phi_j(\mu_1, \mu_2, \lambda) \equiv L_{101}\mu_1\lambda + L_{110}\mu_1\mu_2 + o_3(\lambda, \mu_1, \mu_2) = 0$$

$$\Phi_k(\mu_1, \mu_2, \lambda) \equiv L_{011}\mu_2\lambda + L_{020}\mu_2{}^2 + L_{200}\mu_1{}^2 + o_3(\lambda, \mu_1, \mu_2) = 0$$

$$L_{101} = -{}^1/_2\, a_j{}^2, \qquad L_{011} = -{}^1/_2\, a_k{}^2, \qquad L_{110} = 2L_{200}$$

$$L_{020} = \frac{1}{2}\int_0^\pi \{\varepsilon^2(\nu - 3)a_k{}^3 + 3a_k{}^2b_k\}[P_k{}^1(\cos\xi)]^3\cos\xi\, d\xi \tag{3.14}$$

$$L_{110} = \int_0^\pi \{\varepsilon^2(\nu - 3)a_j{}^2a_k + 2a_ka_jb_j + a_j{}^2b_k\}[P_j{}^1(\cos\xi)]^2 P_k{}^1(\cos\xi)\cos\xi\, d\xi$$

$$ord\ o_3(\lambda, \mu_1, \mu_2) \geqslant 3$$

The quantities a_j, a_k, b_j, b_k are defined in (2.4), (3.2). According to [3] we eliminate μ_1 from (3.14) and solve the equation obtained for μ_2. Then substituting $\mu_2(\lambda)$, we find $\mu_1(\lambda)$. We finally have

$$\mu_1 = \omega_i\lambda + o(\lambda), \quad \mu_2 = \theta_i\lambda + o(\lambda) \qquad (i = 1, 2, 3)$$

$$\omega_1 = 0, \quad \theta_1 = -L_{011}L_{020}{}^{-1}, \quad \theta_2 = \theta_3 = -L_{101}L_{110}{}^{-1}$$

$$\omega_2 = -\omega_3 = [L_{200}L_{101}(L_{011}L_{110} - L_{020}L_{101})]^{1/2}(L_{200}L_{110})^{-1}$$

i.e., three new solutions originate in the half-neighborhoods $(\rho_k - \delta, \rho_k)$ and $(\rho_k, \rho_k + \delta)$ whose asymptotic representation is written as

$$x_i = \omega_i\varphi_j\lambda + \theta_i\varphi_k\lambda + o(\lambda) \qquad (i = 1,2,3) \tag{3.15}$$

The following theorem permits a judgment on the behavior of a shell as the parameter λ changes in the neighborhood of ρ_*

Theorem 4. If $\lambda \in (\rho_*, \rho_* + \delta)$, then the energy functional $E(\rho) < 0$. If $\lambda \in (\rho_* - \delta, \rho_*)$, then $E(\rho) > 0$, while the functional $E(\rho) \equiv 0$ at the trivial solution $x = (0, 0)$. This theorem follows easily from (3.10), (3.11), (3.15) and the form of the energy functional. It can also be shown that the trivial solution for the value $\rho \leqslant \rho_*$ realizes the minimum of the potential energy (the second variation is positive) and is therefore stable. The proof of this fact is carried out by using the reasoning from [11].

Formulas (3.10), (3.11), (3.15) were utilized to construct bifurcation equilibrium modes. The computation of the coefficients of the bifurcation equation in the mentioned formulas was performed on an electronic computer. The integral (3.9) was calculated by recursion formulas which are easily deduced from [12]

$$S_{jk} = \frac{2j-1}{j-1}\,(U_{jk} - V_{jk}) - \frac{j}{j-1}\,S_{j-2,k},\quad S_{0k} = 0$$

$$U_{jk} = 4\pi^{-1}\left[\frac{\Gamma(s-k+{}^1/_2)}{(t-k)!}\right]^2 \frac{(t+1)!\;\Gamma(s-j+{}^3/_2)}{(t-j+1)!\;\Gamma(t+{}^3/_2)}$$

$$V_{jk} = {}^1/_4\pi^{-1}\left[k(k+1)\,\frac{\Gamma(t-k+{}^1/_2)}{(s-k)!}\right]^2 \frac{j(j-1)(s-1)!\;\Gamma(t-j+{}^3/_2)}{(s-j+1)!\;\Gamma(s+{}^3/_2)}$$

$$j = 2i,\; j \leqslant 2k,\; s = k+i,\; t = s-1$$

The computations were performed for values of ε in the interval $1 \times 10^{-4} \leqslant \varepsilon \leqslant 4.3 \times 10^{-2}$. Below, for example, we present values of the bifurcation coefficients for each of three cases. The first two from Theorem 1 and the last from Theorem 3

$\varepsilon =$	0.919×10^{-2}	0.294×10^{-2}	0.841×10^{-3}	0.463×10^{-3}
$L_{11} =$	-0.413188×10^{-4}	-0.427478×10^{-5}	-0.353082×10^{-6}	-0.106969×10^{-6}
$L_{30} =$	0.183134×10^{-7}	0.260063×10^{-9}	0.242242×10^{-11}	0.258120×10^{-12}
$\varepsilon =$	0.113×0^{-1}	0.328×10^{-2}	0.892×10^{-3}	0.483×10^{-3}
$L_{11} =$	-0.617206×10^{-4}	-0.533977×10^{-5}	-0.397177×10^{-6}	-0.116689×10^{-6}
$L_{30} =$	-0.312538×10^{-5}	-0.116177×10^{-6}	-0.336578×10^{-8}	-0.629584×10^{-9}
$\varepsilon =$	0.839×10^{-2}	0.310×10^{-2}	0.817×10^{-3}	0.473×10^{-3}
$L_{101} =$	-0.286944×10^{-4}	-0.533977×10^{-5}	-0.314941×10^{-6}	-0.116689×10^{-6}
$L_{011} =$	-0.413188×10^{-4}	-0.427478×10^{-5}	-0.353082×10^{-6}	-0.106969×10^{-6}
$L_{110} =$	0.232177×10^{-7}	0.686889×10^{-9}	0.419973×10^{-11}	0.575527×10^{-12}
$L_{020} =$	0.183391×10^{-7}	0.259981×10^{-9}	0.242254×10^{-11}	0.258115×10^{-12}

Let us note that values of $\beta'(0)$ and $\psi'(0)$, which can be utilized as initial parameters to find solutions by the Runge-Kutta method, are now easily calculated from (3.10), (3.11), (3.15).

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VARIATIONAL PRINCIPLES OF THE NONLINEAR THEORY OF ELASTICITY. CASE OF SUPERPOSITION OF A SMALL DEFORMATION ON A FINITE DEFORMATION

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L. M. ZUBOV
(Rostov-on-Don)
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General relationships and variational theorems in the theory of small elastic deformations of an elastic solid, superposed on a finite deformation, are presented. A relationship is established between the two modes of the equilibrium equation: in the metric of the undeformed state, and in the metric of an initially deformed state of the solid. A formula is obtained from the potential energy accumulated in an elastic prestressed solid for a small deformation. Variational principles, analogous to the variational principles of the theory of finite deformations [1] and differing from the variational theorems of classical theory of elasticity by the nonsymmetry of the dual tensors, are formulated.

A generalization of the Clapeyron and Betti theorem to the case of small deformations of a prestressed elastic solid is obtained. The formulated variational principles refer, in particular, to the problem of bifurcation of equilibrium of a nonlinearly elastic solid.

Let v be a volume occupied by an elastic solid in an undeformed state, and V° the volume it occupies after deformation caused by mass forces $\mathbf{K}^\circ$ and surface forces $\mathbf{F}^\circ$ ($\mathbf{F}^\circ$ is the vector of forces per unit area of the undeformed solid).

An equilibrium state given by the radius vector of a point of the deformed solid

$$\mathbf{R} = \mathbf{R}^0 + \eta \mathbf{w}$$